# FP3 June 2017 (IAL) (MA)

Q1) $18\left(\frac{e^x+e^{-x}}{2}\right)+14\left(\frac{e^x-e^{-x}}{2}\right)=11+e^x$

$$9e^x+9e^{-x}+7e^x-7e^{-x}-e^x=11$$

$$16e^x+2e^{-x}-e^{x}-11=0$$

$$15e^x+2e^{-x}-11=0$$

$\times e^x$ : $15e^{2x}-11e^x+2=0$

By Quadratic Formula : $e^x=\frac{2}{5}$

$a=15$ $\qquad e^x=\frac{1}{3}$

$b=-11$

$c=2$ $\qquad \therefore \boxed{x=\ln\frac{2}{5}}$

and $\boxed{x=\ln\frac{1}{3}}$

Q2a) $A^T=\begin{pmatrix}-1 & 2 & 1\\ 3 & 0 & -2\\ a & 1 & 1\end{pmatrix}$

b) $AB=\begin{pmatrix}-1 & 3 & a\\ 2 & 0 & 1\\ 1 & -2 & 1\end{pmatrix}\begin{pmatrix}2 & 0 & 4\\ 3 & -2 & 3\\ 1 & 2 & b\end{pmatrix}$

$$=\begin{pmatrix}-2+9+a & -6+2a & 5+ab\\ 5 & 2 & 8+b\\ 2-6+1 & 6 & b-6+4\end{pmatrix}=\begin{pmatrix}a+7 & 2a-6 & 5+ab\\ 5 & 2 & 8+b\\ -3 & 6 & b-2\end{pmatrix}$$

c) $(AB)^T = \begin{pmatrix} a+7 & 5 & -3 \\ 2a-6 & 2 & 6 \\ 5+ab & b+8 & b-2 \end{pmatrix}$

$B^T = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -2 & 2 \\ 4 & 3 & b \end{pmatrix}$

$A^T = \begin{pmatrix} -1 & 2 & 1 \\ 3 & 0 & -2 \\ a & 1 & 1 \end{pmatrix}$

$B^T A^T = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -2 & 2 \\ 4 & 3 & b \end{pmatrix} \begin{pmatrix} -1 & 2 & 1 \\ 3 & 0 & -2 \\ a & 1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 7+a & 4+1 & 2-6+1 \\ -6+2a & 2 & 4+2 \\ (-4+9+ab) & 8+b & b-6+4 \end{pmatrix}$

$= \begin{pmatrix} a+7 & 5 & -3 \\ 2a-6 & 2 & 6 \\ 5+ab & b+8 & b-2 \end{pmatrix} = (AB)^T$

d)

[A and B have dimensions $n \times m$ and $m \times p$]

$\therefore (AB)^T = B^T A^T$

Q3a) $x - y = \operatorname{artanh}\left(\frac{2x}{1+x^2}\right)$

$$\frac{d}{dx}(x-y) = \frac{d}{dx}\left(\operatorname{artanh}\left(\frac{2x}{1+x^2}\right)\right)$$

$$1 - \frac{dy}{dx} = \frac{d}{dx}\left(\operatorname{artanh}\frac{2x}{1+x^2}\right)$$

$\underline{\frac{d}{dx}\left(\operatorname{artanh}\left(\frac{2x}{1+x^2}\right)\right)}$ :

$$\frac{d}{dx}\left(\frac{2x}{1+x^2}\right) = 2(1+x^2)^{-1} - 4x^2(1+x^2)^{-2}$$

$$= \frac{2(1+x^2) - 4x^2}{(1+x^2)^2} = \frac{2-2x^2}{(1+x^2)^2}$$

$$\therefore \frac{d}{dx}\left(\operatorname{artanh}\frac{2x}{1+x^2}\right) = \left[\frac{1}{1 - \left(\frac{2x}{1+x^2}\right)^2}\right] \times \frac{2-2x^2}{(1+x^2)^2}$$

$$= \frac{2-2x^2}{\left[1 - \frac{4x^2}{(1+x^2)^2}\right]\left[(1+x^2)^2\right]}$$

$$= \frac{2-2x^2}{\left[(1+x^2)^2 - 4x^2\right]} = \frac{2(1-x^2)}{x^4 - 2x^2 + 1}$$

$$= \frac{2\cancel{(1-x^2)}}{(1-x^2)^{\cancel{2}}} = \boxed{\frac{2}{1-x^2}} = 1 - \frac{dy}{dx}$$

$\therefore (k = 2)$

b) $\frac{d}{dx}\left[1-\frac{dy}{dx}\right] = \frac{d}{dx}\left[2(1-x^2)^{-1}\right]$

$\cancel{-}\frac{d^2y}{dx^2} = \cancel{-}2(1-x^2)^{-2}(-2x)$.

$\frac{d^2y}{dx^2} = \frac{-4x}{(1-x^2)^2}$

$\frac{d^2y}{dx^2} + \frac{4x}{(1-x^2)^2} = 0$

$x\left(1-\frac{dy}{dx}\right)^2 = x\left(\frac{2}{1-x^2}\right)^2 = \frac{4x}{(1-x^2)^2}$ //

$\therefore \frac{d^2y}{dx^2} + x\left(1-\frac{dy}{dx}\right)^2 = 0$

Q4a) $M-\lambda I = \begin{pmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{pmatrix}$

$\det(M-\lambda I) = 1-\lambda \begin{vmatrix} 5-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 3 & 1-\lambda \end{vmatrix} + 3 \begin{vmatrix} 1 & 5-\lambda \\ 3 & 1 \end{vmatrix}$

$\det(M-\lambda I) = 1-\lambda\left[(5-\lambda)(1-\lambda)-1\right] - \left[1-\lambda-3\right]$
$+ 3\left[1-(5-\lambda)(3)\right] = 0$ //

$\det(M-\lambda I) = (1-2\lambda+\lambda^2)(5-\lambda) + \lambda - 1 + 2 + \lambda$
$+ 3 - 9(5-\lambda) = 0$

$\det(M-\lambda I) = 5 - \cancel{10\lambda} + 5\lambda^2 - \cancel{\lambda} + 2\lambda^2 - \lambda^3 + \cancel{2\lambda} + 1 + 3 - 45 + \cancel{9\lambda}$
$= 0$

$\Rightarrow -\lambda^3 + 7\lambda^2 - 36 = 0$

$(\lambda = 6)$: $-216 + 252 - 36 = 0$ // $\therefore \lambda = 6$ is an eigenvalue.

$$\begin{array}{r l}
 & -\lambda^2 + \lambda + 6 \\
\lambda - 6 & \overline{\big) -\lambda^3 + 7\lambda^2 + 0\lambda - 36} \\
 & -\lambda^3 + 6\lambda^2 \\
 & \overline{0 \quad \lambda^2 + 0\lambda} \\
 & \lambda^2 - 6\lambda \\
 & \overline{0 + 6\lambda - 36} \\
 & 6\lambda - 36 \\
 & \overline{0 \quad 0}
\end{array}$$
//

$\therefore -(\lambda-6)(\lambda^2-\lambda-6) = 0$

$\hookrightarrow (\lambda-3)(\lambda+2) = 0$ //

$\therefore \boxed{\lambda = 3}, \boxed{\lambda = -2}$

are the other eigenvalues.

b) $\underline{Ax = \lambda x}$ : $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6x \\ 6y \\ 6z \end{pmatrix}$ — ① — ② — ③

① : $x + y + 3z = 6x$

② : $x + 5y + z = 6y$

③ : $z + y + 3x = 6z$

b cont.) (2) : $y = x + z$.

let $x = 1$ : $y = z + 1$

(1) : $1 + z + 1 + 3z = 6$

$4z = 4$

$\therefore z = 1$

and $y = 2$

$\therefore$ a corresponding eigenvector is $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

normalised : $\dfrac{1}{\sqrt{1^2+2^2+1^2}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$= \boxed{\dfrac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}$

5a) $I_n = \int [\operatorname{cosec}^{n-2}x][\operatorname{cosec}^2 x]\,dx$

$= \int [1+\cot^2 x][\operatorname{cosec}^{n-2}x]\,dx$

$= \int [\operatorname{cosec}^{n-2}x]\,dx + \int [\cot^2 x \operatorname{cosec}^{n-2}x]\,dx$

$\therefore I_n = I_{n-2} + \int [\operatorname{cosec}x \cot x][\cot x][\operatorname{cosec}^{n-3}x]\,dx$

$I_n = I_{n-2} - \int [-\operatorname{cosec}x\cot x][\operatorname{cosec}^{n-3}x][\cot x]\,dx$

$\frac{d}{dx}(\operatorname{cosec}x) = -\operatorname{cosec}x\cot x$

$\therefore I_n =$

$\frac{dv}{dx} = [\operatorname{cosec}^{n-3}x][-\operatorname{cosec}x\cot x]$

(By pattern) $v = \frac{\operatorname{cosec}^{n-2}x}{n-2}$

$u = \cot x$
$u' = -\operatorname{cosec}^2 x$

$\Rightarrow I_n = I_{n-2} - \left[\frac{\cot x \operatorname{cosec}^{n-2}x}{n-2} + \frac{1}{n-2}\int [\operatorname{cosec}^n x]\,dx\right]$

$\Rightarrow I_n = I_{n-2} - \frac{\cot x \operatorname{cosec}^{n-2}x}{n-2} - \frac{1}{n-2}(I_n)$

$\Rightarrow \left(1+\frac{1}{n-2}\right)I_n = I_{n-2} - \frac{\cot x \operatorname{cosec}^{n-2}x}{n-2}$

$$\Rightarrow \left(\frac{n-1}{n-2}\right) I_n = I_{n-2} - \frac{\cot x \operatorname{cosec}^{n-2} x}{n-2}$$

$$\times\left(\frac{n-2}{n-1}\right): \quad I_n = \frac{n-2}{n-1} I_{n-2} - \frac{1}{n-1}\cot x \operatorname{cosec}^{n-2} x$$

b) $I_4 = \frac{4-2}{4-1} I_2 - \frac{1}{3}\cot x \operatorname{cosec}^2 x$

$(1+\cot^2 x = \operatorname{cosec}^2 x)$

$$I_4 = \frac{2}{3} I_2 - \frac{1}{3}\cot x\left(1+\cot^2 x\right)$$

and $I_2 = \int(\operatorname{cosec}^2 x)\,dx = -\cot x \quad (+c)$

$$\therefore I_4 = \frac{2}{3}(-\cot x) - \frac{1}{3}\cot x - \frac{1}{3}\cot^3 x \quad (+c)$$

$$\Rightarrow I_4 = -\cot x - \frac{1}{3}\cot^3 x + c$$

Q6a) $P(a\sec\theta, b\tan\theta)$

$x = a\sec\theta \rightarrow \frac{dx}{d\theta} = a\sec\theta\tan\theta$

$y = b\tan\theta \rightarrow \frac{dy}{d\theta} = b\sec^2\theta$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b\sec\theta}{a\tan\theta} = \frac{\frac{b}{\cos\theta}}{\frac{a\sin\theta}{\cos\theta}}$$

$$= \frac{b}{a\sin\theta} = m$$

$$\therefore\ y - b\tan\theta = \frac{b}{a\sin\theta}(x - a\sec\theta)$$

$$y = \frac{bx}{a\sin\theta} - \frac{b}{\sin\theta\cos\theta} + \frac{b\sin\theta}{\cos\theta}$$

$$\boxed{\times\ \frac{a\sin\theta}{\cos\theta}}\ :\quad ay\tan\theta = \frac{bx}{\cos^2\theta} - \frac{ab}{\cos^2\theta} + \frac{ab\sin^2\theta}{\cos^2\theta}$$

$$bx\sec\theta - ay\tan\theta = \frac{ab(1-\sin^2\theta)}{\cos^2\theta}$$

$$\therefore\ bx\sec\theta - ay\tan\theta = ab$$

b) $F: (ae, 0)$ $\qquad b^2 = a^2(1-e^2)$

we need to show the gradient of $l = 1$.

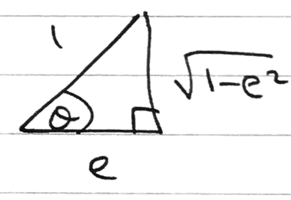
$\left(m_l = \frac{b}{a\sin\theta}\right)$.

$y=0, x=ae$: $\quad bae\sec\theta = ab$

$$ab(e\sec\theta - 1) = 0$$

$$(\sec\theta)e = 1$$

$$\therefore\ \cos\theta = e$$




$$\therefore\ \sin\theta = \sqrt{1-e^2}.$$

and $b^2 = a^2(1-e^2) = a^2(1-\cos^2\theta)$

$$= a^2\sin^2\theta$$

$$\therefore\ b = a\sin\theta.$$

now $m_l = \frac{b}{a\sin\theta} = \frac{a\sin\theta}{a\sin\theta} = 1$ $\quad\therefore$ parallel to $y=x$.

7a) $\int \frac{5+x}{\sqrt{4-3x^2}}\,dx$ ~ let $x = \frac{2}{\sqrt{3}}\sin\theta$

$\frac{dx}{d\theta} = \frac{2}{\sqrt{3}}\cos\theta$

$dx = \frac{2}{\sqrt{3}}\cos\theta\,d\theta$.

$\Rightarrow \int \frac{\left(5+\frac{2}{\sqrt{3}}\sin\theta\right)}{\sqrt{4-3\left(\frac{4}{3}\sin^2\theta\right)}}\,d\theta \times \frac{2}{\sqrt{3}}\cos\theta$

$\Rightarrow \int \frac{\left(5+\frac{2}{\sqrt{3}}\sin\theta\right)\left(\frac{2}{\sqrt{3}}\cos\theta\right)}{\sqrt{4(1-\sin^2\theta)}}\,d\theta$

$\Rightarrow \frac{2}{\sqrt{3}}\int \left[\frac{\left(5+\frac{2}{\sqrt{3}}\sin\theta\right)\cancel{\cos\theta}}{2\cancel{\cos\theta}}\right] d\theta$

$\Rightarrow \frac{1}{\sqrt{3}}\int \left[5+\frac{2}{\sqrt{3}}\sin\theta\right] d\theta = \frac{1}{\sqrt{3}}\left[5\theta - \frac{2}{\sqrt{3}}\cos\theta\right] + c$

$= \frac{5}{\sqrt{3}}\theta - \frac{2}{3}\cos\theta + c$.

$\frac{\sqrt{3}}{2}x = \sin\theta \quad \therefore \theta = \arcsin\left(\frac{\sqrt{3}}{2}x\right)$

$= \frac{5}{\sqrt{3}}\arcsin\left(\frac{\sqrt{3}}{2}x\right) - \frac{2}{3}\cos\left[\arcsin\left(\frac{\sqrt{3}}{2}x\right)\right] + c$

b) $$\left[\frac{5}{\sqrt{3}}\arcsin\left(\frac{\sqrt{3}}{2}x\right) - \frac{2}{3}\cos\left(\operatorname{arcsin}\left(\frac{\sqrt{3}}{2}x\right)\right)\right]_0^1$$

$$= \left[\frac{5}{\sqrt{3}} \times \frac{\pi}{3} - \frac{2}{3} \times \frac{1}{2}\right] - \left[-\frac{2}{3}\right]$$

$$= \frac{5\pi}{3\sqrt{3}} - \frac{1}{3} + \frac{2}{3} = \boxed{\frac{5\pi\sqrt{3}}{9} + \frac{1}{3}}$$

Q8a) $x = \theta - \sin\theta \qquad y = 1 - \cos\theta$

$$\frac{dx}{d\theta} = 1 - \cos\theta \qquad \frac{dy}{d\theta} = \sin\theta$$

$$\text{Area} = 2\pi\int_0^{2\pi} y\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= 2\pi\int_0^{2\pi}\left[(1-\cos\theta)\sqrt{(1-\cos\theta)^2 + \sin^2\theta}\right] d\theta$$

$$= 2\pi\int_0^{2\pi}(1-\cos\theta)\sqrt{1 - 2\cos\theta + \sin^2\theta + \cos^2\theta}\, d\theta$$

$$= 2\pi\int_0^{2\pi}\left[(1-\cos\theta)\sqrt{2 - 2\cos\theta}\right] d\theta$$

$$= 2\pi\sqrt{2}\int_0^{2\pi}(1-\cos\theta) \times (1-\cos\theta)^{\frac{1}{2}}\, d\theta = 2\pi\sqrt{2}\int_0^{2\pi}(1-\cos\theta)^{3/2}\, d\theta$$

b) $S = 2\pi\sqrt{2}\int_0^{2\pi} (1-\cos\theta)^{3/2}\, d\theta \qquad \cos\theta = 1 - 2\sin^2\left(\frac{\theta}{2}\right)$

$$= 2\pi\sqrt{2}\int_0^{2\pi} \left(1-1+2\sin^2\frac{\theta}{2}\right)^{3/2} d\theta$$

$$= 2\pi\sqrt{2}\int_0^{2\pi} \left(2\sin^2\frac{\theta}{2}\right)^{\frac{3}{2}} d\theta = 2\pi\sqrt{2}\int_0^{2\pi} \left(\sqrt{2}\sin\frac{\theta}{2}\right)^3 d\theta$$

$$= 2\pi\sqrt{2}\int_0^{2\pi} 2\sqrt{2}\sin^3\left(\frac{\theta}{2}\right) d\theta = 8\pi\int_0^{2\pi} \sin^3\frac{\theta}{2}\, d\theta$$

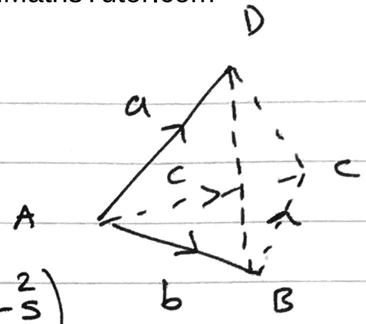
$$= 8\pi\int_0^{2\pi} \left(1-\cos^2\frac{\theta}{2}\right)\sin\frac{\theta}{2}\, d\theta = 8\pi\int_0^{2\pi} \left[\sin\frac{\theta}{2} - \sin\frac{\theta}{2}\cos^2\frac{\theta}{2}\right] d\theta$$

$$= 8\pi\left[-2\cos\left(\frac{\theta}{2}\right) + \frac{2\cos^3\left(\frac{\theta}{2}\right)}{3}\right]_0^{2\pi}$$ By pattern.

$$= 8\pi\left[2 + \frac{1}{6}(-4)\right] - 8\pi\left[-2 + \frac{4}{6}\right]$$

$$= \boxed{64\pi/3}$$

There are various approaches you could take for this Q.




9a) $\text{Vol.} = \left|\frac{1}{6}a\cdot(b\times c)\right|$

$$\overrightarrow{AB} = \begin{pmatrix}1\\0\\3\end{pmatrix} - \begin{pmatrix}-1\\5\\1\end{pmatrix} = \begin{pmatrix}2\\-5\\2\end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix}2\\-1\\2\end{pmatrix} - \begin{pmatrix}-1\\5\\1\end{pmatrix} = \begin{pmatrix}3\\-6\\1\end{pmatrix}$$

$$\overrightarrow{AD} = \begin{pmatrix}3\\6\\-1\end{pmatrix} - \begin{pmatrix}-1\\5\\1\end{pmatrix} = \begin{pmatrix}4\\1\\-2\end{pmatrix}$$

$$\overrightarrow{AC}\times\overrightarrow{AD} = \begin{pmatrix}3\\-6\\1\end{pmatrix}\times\begin{pmatrix}4\\1\\-2\end{pmatrix} = \begin{pmatrix}11\\10\\27\end{pmatrix}$$

$$(\overrightarrow{AB})\cdot(\overrightarrow{AC}\times\overrightarrow{AD}) = \begin{pmatrix}2\\-5\\2\end{pmatrix}\cdot\begin{pmatrix}11\\10\\27\end{pmatrix} = 26 \,//$$

$$\therefore \text{Volume} = \frac{1}{6}\times 26 = \boxed{\frac{13}{3}} \text{ units}^3$$

b) $$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{pmatrix}2\\-5\\2\end{pmatrix}\times\begin{pmatrix}3\\-6\\1\end{pmatrix} = \begin{pmatrix}7\\4\\3\end{pmatrix}$$

$$\therefore r\cdot\begin{pmatrix}7\\4\\3\end{pmatrix} = \begin{pmatrix}7\\4\\3\end{pmatrix}\cdot\begin{pmatrix}-1\\5\\1\end{pmatrix} = 16 \,//$$

So $\boxed{7x + 4y + 3z = 16}$

c) $\overrightarrow{DT}$ is parallel to the normal of $\Pi$ and passes through D. $\therefore$ eqn of line through D and T is: $r = \begin{pmatrix}3\\6\\-1\end{pmatrix} + \lambda\begin{pmatrix}7\\4\\3\end{pmatrix}$

<u>Sub into eqn for $\Pi$</u> : $\begin{pmatrix} 3+7\lambda \\ 6+4\lambda \\ 3\lambda - 1 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ 4 \\ 3 \end{pmatrix} = 16$

$$21 + 49\lambda + 24 + 16\lambda + 9\lambda - 3 = 16$$

$$74\lambda = -26$$

$$\lambda = -13/37 \;//$$

$\cancel{\overrightarrow{OX}}$ $\overrightarrow{OT} = \begin{pmatrix} 3+7\left(-13/37\right) \\ 6+4\left(-13/37\right) \\ 3\left(\frac{-13}{37}\right) - 1 \end{pmatrix} = \boxed{\begin{pmatrix} 20/37 \\ 170/37 \\ -76/37 \end{pmatrix}}$